THEOREMS ON THE ASYMPTOTIC STABILITY OF SOLUTIONS OF CERTAIN THIRD ORDER DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS CHARACTERISTICS

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A method of stabilizing nonlinear third order control systems is considered. The property of stability is achieved by increasing certain system parameters. Exactly as in [1], points in the phase space are first transferred to a certain surface and then move over this surface to the origin of the coordinates in a slip regime. However, in contrast to [1], the introduction of additional changed parameters permits to insure a slip regime for any motion throughout the whole time starting with a certain instant, and this affords the possibility of obtaining the property of asymptotic stability of the zero solution.

Correction of a linear system by using many changed parameters was studied earlier [2], however, only the slip conditions are obtained herein and stability questions are not considered. Let us note that in the linear case the possibility of achieving the asymptotic stability proprty for a third order system has been established on the basis of an idea in [1] by V.P.Baranovskii.

1. Let us consider the differential equation

$$x''' + F(x, x', x'', t) + (K|x| + K_1|x'|)\operatorname{sign}(x'' - \varphi(x, x')) = 0 \tag{1.1}$$

Here K and K_1 are positive parameters $F(x,x',x'',t)$ and $\varphi(x,x')$ are contunuous functions of their arguments for all values of x, x', x'' and $t \geqslant 0$.

Equation (1.1) is equivalent to the system

$$x' = y, \quad y' = z, \quad z' = -F(x, y, z, t) - (K|x| + K_1|y|)\operatorname{sign}(z - \varphi(x, y)) \tag{1.2}$$

Let us assume compliance with conditions

a) $|F(x, y, z, t)| \leqslant a|x| + b|y| + c|z|$ for any values of x, y, z and $t \geqslant 0$. Here a, b and c are non-negative constants.

b) The function $\varphi(x,y)$ is defined everywhere and is continuously differentiable with respect to x and y, where positive numbers M and N exist such that

$$\left|\frac{\partial\varphi}{\partial x}\right| \leqslant M, \qquad \left|\frac{\partial\varphi}{\partial y}\right| \leqslant N$$

c) $x\varphi(x, 0) < 0$ for $x \neq 0$

$$y[\varphi(x, y) - \varphi(x, 0)] < 0 \quad \text{for} \quad y \neq 0, \qquad \int_{\mp\infty}^{0} \varphi(x, 0)\, dx = \infty$$

It is easy to see that the right-hand side of the third equation of (1.2) undergoes a discontinuity on the surface S which is given by Equation $z = \varphi(x, y)$. Let $r(x, y, z) = z - \varphi(x, y)$. If

$$\lim_{r\to-0} \frac{dr}{dt} > 0, \qquad \lim_{r\to+0} \frac{dr}{dt} < 0$$

where the derivative dr/dt is taken by virtue of the equations of the system (1.2), then a motion is defined on the surface S which is described by the system of differential equations

$$x^{\cdot} = y,\ y^{\cdot} = \varphi(x, y) \tag{1.3}$$

It is accepted to say in this case that the system (1.3) describes the slip regime.

Let us note that the condition (a) guarantees continuation [3] (p.16) of the motion of the system (1.2) at least up to the time when the point hits the surface S. If all the points of a certain domain G of the phase space hit the surface S during their motion, and then move over S to the origin with rising t by virtue of the system (1.3), then the required property of asymptotic stability of the zero solution is obtained.

T h e o r e m 1.1 . Let the functions $F(x,y,z,t)$ and $\varphi(x,y)$ satisfy conditions (a), (b) and (c) and let the parameter K_1 be fixed and selected according to the inequality

$$K_1 \geqslant b + M + cN + N^2 \tag{1.4}$$

and let the bounded domain G of the phase space be given. A positive number K_0 can be indicated such that for $K > K_0$ the zero solution of the system (1.2) will be asymptotically stable, where the domain G will lie in a region of attraction of the origin.

P r o o f . Let us first show that by increasing K, the slip regime of the whole surface S can be obtained. Actually, taking the derivative of the function $r(x,y,z)$ by virtue of the system (1.2), we have

$$\frac{dr}{dt} = \Phi(x, y, r, t) - (K|x| + K_1|y|)\,\mathrm{sign}\, r \tag{1.5}$$

Here

$$\Phi(x, y, r, t) = -F(x, y, r + \varphi(x, y), t) - \frac{\partial\varphi}{\partial x} y - \frac{\partial\varphi}{\partial y}(r + \varphi(x, y))$$

The functions $F(x,y,z,t)$ and $\varphi(x,y)$ satisfy the relations (a) and (b), hence the function $\Phi(x,y,r,t)$ also satisfies an inequality analogous to (a)

$$|\Phi(x, y, r, t)| \leqslant A|x| + B|y| + C|r| \quad .$$

where

$$A = a + cM + NM, \qquad B = b + M + cN + N^2, \qquad C = c + N$$

Let us calculate the limit values of the derivative dr/dt as the representative point of the system (1.2) approaches the surface S

$$\lim_{r\to -0}\frac{dr}{dt} = \Phi(x, y, 0, t) + K|x| + K_1|y| > (K - A)|x| + (K_1 - B)|y|$$

$$\lim_{r\to +0}\frac{dr}{dt} = \Phi(x, y, 0, t) - K|x| - K_1|y| < (A - K)|x| + (B - K_1)|y|$$

for all values of x and y.

In order to guarantee the slip regime on the whole surface $r = z - \varphi(x,y) = 0$ it is evidently sufficient to require compliance with the inequalities $K > A$ and $K_1 \geqslant B$. Let us now show that for any bounded domain G of the phase space, a value $K_0 > 0$ can be selected such that for $K > K_0$ any point M of the domain G moving along the trajectory of the system (1.2) with rising t will hit the surface S.

Exactly as in [1], let us make a change of variables in the system (1.2)

$$X = x, \quad Y = \rho y, \quad Z = \rho^2 z, \quad t = \rho\tau, \quad K^{-1/3}$$

The new system will have the form

$$\frac{dX}{d\tau} = Y, \quad \frac{dY}{d\tau} = Z \qquad \left(R = Z - \rho^2\varphi\left(X, \frac{Y}{\rho}\right)\right) \tag{1.6}$$

$$\frac{dZ}{d\tau} = -|X|\operatorname{sign} R - \rho^2 K_1|Y|\operatorname{sign} R - \rho^3 F\left(X, \frac{Y}{\rho}, \frac{Z}{\rho^2}, \rho\tau\right)$$

For large values of K the quantity ρ plays the part of a small parameter, hence the simplified system is written for $\rho = 0$ as

$$\frac{dX}{d\tau} = Y, \quad \frac{dY}{d\tau} = Z, \quad \frac{dZ}{d\tau} = -|X|\operatorname{sign} R \tag{1.7}$$

Boundedness of $\rho\varphi(X, Y/\rho)$ for small ρ and bounded X and Y follows from condition (b).

In [1, (2.3)] the system (1.7) has been investigated and it has been shown that any point of the phase space moving with rising t along the trajectory of the system (1.7) will hit the surface S at a finite time. Only points in phase space which lie on the integral line $X = -Y = Z$ are the exception; they approach the origin of the coordinates asymptotically.

Boundedness of the quantity $\rho^2 F(X, \rho^{-1}Y, \rho^{-2}Z, \rho\tau)$ in the domain G for small values of ρ follows from condition (a). Selecting ρ to be sufficiently small and using the well-known consideration resulting from the property of continuity of the solutions in a parameter, we arrive at the conclusion that all points of the domain G (with the exception of a narrow enough vicinity along the line $X = -Y = Z$) which move by virtue of the system (1.6), will hit the surface S in a finite time interval and points of the vicinity mentioned will either emerge from this vicinity as time rises and hit S, or will remain within it and, therefore, will also hit S in a finite or infinite time interval.

Having fallen on the surface S, the representative point will move over it by virtue of the system (1.3). Conditions (c) guarantee asymptotic stability of the zero solution of this system according to [4]. Hence, it has been shown that any point of the domain G will approach the origin asymptotically.

Since the definition of asymptotic stability includes the requirement of compliance with the usual property of stability in the Liapunov sence, it is still necessary to show now that the points of a sufficiently small neighborhood of the origin of the coordinates do not emerge beyond the limits of the given neighborhood.

To do this, let us replace the coordinate Z in the system (1.6) by the new coordinate $R = Z - \rho^2\varphi(X, Y/\rho)$. The new system will have the form

$$\frac{dX}{d\tau} = Y, \quad \frac{dY}{d\tau} = R + \rho\varphi_1(X, Y, \rho), \quad \frac{dZ}{d\tau} = -|X|\operatorname{sign} R + \rho F_1(X, Y, Z, \rho, \tau) \tag{1.8}$$

where $\varphi_1(X, Y, \rho)$ and $F_1(X, Y, Z, \rho, \tau)$ also satisfy conditions of type (a) and (b).

Considering the half-space $R > 0$ for definiteness, let us write the simplified system as

$$\frac{dX}{d\tau} = Y, \qquad \frac{dY}{d\tau} = R, \qquad \frac{dR}{d\tau} = -|X| \tag{1.9}$$

Evidently the positive quantity R decreases and Y increases along the trajectories of the system (1.9). Let is first show that for an arbitrary initial point $M_0(X_0, Y_0, R_0)$ the following lemma is valid:

Lemma . The point $M(\tau)$ moving along the trajectory of the system (1.9) can not be in the domain $|X| > \delta$, $R \neq 0$ more than R_0/δ time units.

Actually, we have $R < R_0 - \delta\tau$ from the last equation of (1.9), where τ is measured from the instant at which compliance with the inequality $|X| > \delta$ starts. If $\tau_1 > R_0/\delta$ then $R(\tau_1)$ will be negative, which may not be since the $R = 0$ plane is the slip plane.

Hence, within the time interval $[0, \tau_1]$ the point $M(\tau)$ either falls in the domain $|X| < \delta$ or hits the plane $R = 0$.

Let us note that if $X < -\delta$ and $Y < 0$ or $X > \delta$ and $Y > 0$ along the trajectory, then the point $M(\tau)$ can hit only the $R = 0$ plane. It hence follows that this point may be in the domain $|X| < \delta$ not more than twice.

Now, let the initial point M_0 lie in the domain $|X_0| < \delta$, $|Y_0| < \delta$, $0 < R_0 < \delta$. Let us estimate the coordinates $X(\tau)$, $Y(\tau)$ and $R(\tau)$ of the point $M(\tau)$ moving from the point M_0 along the trajectory of the system (1.9) up to the time of meeting the plane $R = 0$. Let us first show that the inequality

$$-\delta < Y(\tau) < 4\delta \tag{1.10}$$

holds.

Let us assume that $Y_0 > 0$. Since $X(\tau)$ and $Y(\tau)$ may only rise, then let τ_0 denote the time when $X(\tau_0) = \delta$ and let τ_0' denote the time when $Y(\tau_0') = \delta$. We let τ_1 denote the time when the point $M(\tau)$ meets the $R = 0$ plane.

According to Lemma, if $\tau_1 = \infty$ then $|X(\tau)| < \delta$ for all positive values of τ, if also $\tau_0' = \infty$, then the inequality (1.10) will hold. Two cases are possible. In the first case, let us assume that $\tau_0 \leqslant \tau_0'$.

According to (1.9), we have $Y(\tau) - Y(\tau_0') \leqslant R_0(\tau - \tau_0')$ for $\tau \geqslant \tau_0'$, but since $\tau - \tau_0' < \tau_1 - \tau_0' < 1$, according to Lemma, we then obtain $Y(\tau) < 2\delta$, if $\tau_0' \leqslant \tau \leqslant \tau_1$.

In the second case let us assume that $\tau_0' < \tau_0$ and let us estimate the difference $\tau_0 - \tau_0'$. According to (1.9) we have

$$\delta < Y(\tau) < R_0(\tau - \tau_0') + \delta \quad \text{when } \tau > \tau_0' \tag{1.11}$$

The left-hand side of this in quality yields the estimate

$$\delta(\tau_0 - \tau_0') + X(\tau_0') < X(\tau_0),$$

from which $\tau_0 - \tau_0' < 2$ follows. The right-hand side of the same inequality (1.11) yields the estimate $Y(\tau) < \delta(\tau - \tau_0 + \tau_0 - \tau_0') + \delta < 4\delta$, since according to Lemma $\tau - \tau_0 < 1$.

Now let us consider the case when $Y_0 < 0$. If $Y(\tau)$ does not change sign, then (1.10) is satisfied. If $Y(\tau)$ changes sign at the time τ_3, then two cases are again possible. In the case $|X(\tau_3)| < \delta$ for $\tau > \tau_3$ the point $M(\tau)$ falls under the conditions which have been considered above. If however $X(\tau_3) < -\delta$, then according to Lemma the point $M(\tau)$ can not be outside the strip $|X| < \delta$ more than one time unit, but $Y(\tau)$ may increase but not more that δ within this time and, hence, the point $M(\tau)$ again falls into the domain $|X| < \delta$, $|Y| < \delta$ and the quantity $Y(\tau)$ will be positive and, therefore, will satisfy the estimate (1.10).

Let us now show that until the point $M(\tau)$ hits the $R = 0$ plane, the inequality

$$-2\delta < X(\tau) < 5\delta \tag{1.12}$$

will hold.

Indeed, if the point $M(\tau)$ emerges into the domain $X > \delta$, it will not return to the strip $|X| < \delta$ and it must hit the $R = 0$ plane. But in this case, $X(\tau) - X(\tau_0) < Y(\tau)(\tau - \tau_0)$ for $\tau_0 \leqslant \tau \leqslant \tau_1$. follows from (1.9)

It follows from Lemma that $\tau - \tau_0 < 1$ and from inequality (1.10) we have $0 < Y(\tau) < 5\delta$. Hence, $\delta < X(\tau) < 5\delta$.

If the point $M(\tau)$ emerges into the domain $X < -\delta$ at the time τ_4 then it will either hit the $R = 0$ plane without returning to the strip $|X| < \delta$ or it will return to this strip. In the first case we obtain

$$-\delta(\tau - \tau_4) < X(\tau) - X(\tau_4),$$

from which $X(\tau) > -2\delta$ follows. In the second case the point $M(\tau)$ returns to the domain $|X| < \delta$, $|Y| < \delta$ and the situation already considered will hold. Let us note that the minimum value of $X(\tau_3)$ in the second case will satisfy the inequality $X(\tau_3) > -2\delta$. The inequalities (1.10) and (1.12) together with the inequality $|R(\tau)| < \delta$ prove the asymptotic stability for a simplified system only. It follows from the discussion presented above that the point $M(\tau)$ can be outside of the domain $|X| < \delta$, $|Y| < \delta$ not more than three units of time. Having used the known estimates of the deviation of the solution [5], it can be shown that the same estimates with the accuracy of small orders of $\rho\delta$ hold for trajectories of the system (1.8) starting on the boundary of the mentioned domain.

Since the dynamical system on the $R = 0$ plane has the asymptotic stability property and all the solutions have the property $x(t) \to 0$, $y(t) \to 0$ and $z(t) \to 0$ as $t \to \infty$ according to what was proved above, then the asymptotic stability of the zero solution of the system (1.2) is now obvious.

N o t e 1.1. Let $x_0 > 0$, $y_0 \geqslant 0$ and $z_0 \geqslant \varphi(x_0, y_0)$. Using the result of Lemma, it is easy to obtain an estimate of the time of hit T of the point $M(x_0, y_0, z_0)$ onto the slip surface S for the system (1.2). This estimate has the form

$$T \leqslant \frac{z_0 - \varphi(x_0, y_0)}{Kx_0}(1 + O(\rho))$$

where $O(\rho)$ denotes the order of magnitude of the smallness of ρ.

If x_0, y_0 and z_0 are numbers of arbitrary sign, the total residence time of the representative point in the domain $|X| > \delta$, $z \neq \varphi(x,y)$ will not exceed

$$T_1 = 3\,\frac{|z_0 - \varphi(x_0, y_0)|}{K\delta}(1 + O(\rho))$$

N o t e 1.2. The method of obtaining the inequalities (1.10) and (1.12) permits replacement of the constraint of compliance with conditions (a), (b) and (c) in the whole phase space by the constraint of satisfying them in a certain bounded domain G_1. The proposed method may be used to estimate the domain G_1.

2. Let us consider the differential equation

$$x^{\cdot\cdot\cdot} + F(x, x^{\cdot}, x^{\cdot\cdot}) + (K|x| + K_1|x^{\cdot}| + K_2|x^{\cdot\cdot}|)\operatorname{sign}(x^{\cdot\cdot} - \varphi(x, x^{\cdot})) = 0 \tag{2.1}$$

where the functions $F(x, x^{\cdot}, x^{\cdot\cdot})$ and $\varphi(x, x^{\cdot})$ again satisfy conditions (a), (b) and (c). In contrast to the previous case, let us assume however, that the function $F(x, x^{\cdot}, x^{\cdot\cdot})$ is explicitly independent of t.

Equation (2.1) is equivalent to the system (2.2)

$$x^{\cdot} = y,\ y^{\cdot} = z,\ z^{\cdot} = -F(x, y, z) - (K|x| + K_1|y| + K_2|z|)\operatorname{sign}(z - \varphi(x, y))$$

T h e o r e m 2.1 . If the functions $F(x, y, z)$ and $\varphi(x, y)$ satisfy the conditions (a), (b) and (c) and the parameters K, K_1 and K_2 are chosen according to the inequalities

$$K \geqslant a, \quad K_1 \geqslant b + M + 1, \quad K_2 \geqslant c + N \tag{2.3}$$

then the zero solution of the system (2.2) will be asymptotically stable in the large.

P r o o f . Let us introduce the new coordinate $r = z - \varphi(x, y)$, the system (2.2) in the new x, y and r coordinates will be

$$x^{\cdot} = y \qquad y^{\cdot} = r + \varphi(x, y) \tag{2.4}$$

$$r^{\cdot} = -F(x, y, r + \varphi(x, y)) - \frac{\partial \varphi}{\partial x} y - \frac{\partial \psi}{\partial y}(r + \varphi(x, y)) -$$
$$-(K|x| + K_1|y| + K_2|r + \varphi(x, y)|) \operatorname{sign} r$$

Let us consider the Liapunov function

$$v = r^2 + y^2 - 2\int_0^x \varphi(x, 0)\, dx \tag{2.5}$$

It follows from condition (c) that the function v will be positive definite and infinitely large [6].

Computing the derivative of the function v, by virtue of (2.4) we obtain

$$\frac{dv}{dt} = 2y\,[\varphi(x, y) - \varphi(x, 0)] - 2|r|\,[K|x| + K_1|y| + K_2|r + \varphi(x, y)|] +$$
$$+ 2r\left[-F(x, y, r + \varphi(x, y)) - \frac{\partial \varphi}{\partial x} y - \frac{\partial \varphi}{\partial y}(r + \varphi(x, y)) + y\right]$$

Taking account of conditions (a) and (b), we obtain

$$\frac{dv}{dt} \leqslant 2y\,[\varphi(x, y) - \varphi(x, 0)] -$$
$$-2|r|\,[(K - a)|x| + (K_1 - b - M - 1)|y| + (K_2 - c - N)|r + \varphi(x, y)|]$$

Since the relationships (b) and (2.3) are satisfied by the conditions of the theorem, it then follows from the last inequality that the derivative of the function v taken by virtue of the system (2.4), will be a negative function, vanishing on the x-axis. Evidently there are no entire trajectories of the system (1.2) on the x-axis with the exception of the singular point $O(0,0,0)$; moreover, the function v is infinitely large. Hence, Theorem 4 of [6] may be applied, which indeed completes the proof of our theorem.

As regards considerations relative to the qualitative disposition of the trajectories of the system (1.2), it is easy to see that the surface $z = \varphi(x, y)$ will be slip surface at all its points. This is confirmed by computations similar to those made in the proof of Theorem 1. Hence, the representative point of the system (2.2) either approaches asymptotically the origin directly as $t \to \infty$ or it hits first the slip surface after a certain time and then also approaches the origin asymptotically by having moved over the surface.

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